God's Scales

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Abstract

This paper reveals a hidden symmetry between ancient Pythagorean tuning and modern equal temperaments. By "wrapping" perfect fifths around the octave or tritave, we generate scales that—though not perfectly closed—align astonishingly closely with specific equal divisions of pitch space. Crucially, these large equal temperaments can be *reduced* to familiar small-scale systems (7, 11, 12, or 19 notes) using simple, repeating skip patterns: a sequence of steps like $4, 9, 9, 9$ or its refined version $4, 5, 4, 5, \ldots$. Most profoundly, this reduction works not just for octave- or tritave-based temperaments, but for *any* high-resolution temperament designed to make a single interval—fifth, fourth, major second, etc.—perfectly exact. The result is a family of scales where one interval is mathematically pure, and all others deviate by less than 2 cents: imperceptible to the human ear. No prior musical knowledge is assumed—only curiosity.

1 Introduction: What is a Musical Scale?

Imagine a piano keyboard. Press any key, then press the key twelve steps higher—you hear the same note, but "higher." This repeating distance is called the **octave**, corresponding to a frequency ratio of exactly 2:1. A **scale** is simply a selection of notes within one octave (or another repeating interval) that we use to make music.

For over 2,500 years, musicians and mathematicians have sought "perfect" scales—sets of notes where the intervals (distance between notes) correspond to simple whole-number ratios, like 3:2 (a perfect fifth) or 4:3 (a perfect fourth). The ancient Greek philosopher Pythagoras discovered that by stacking perfect fifths (3/2), he could generate a useful set of notes. This is the origin of **Pythagorean tuning**.

But there's a problem: stacking twelve perfect fifths does *not* land exactly on seven octaves. The mismatch—called the **Pythagorean comma**—means the scale never perfectly "closes." Yet, as we'll see, this very imperfection points toward a deeper harmony.

2 Pythagorean Wrapping: Building Scales from Fifths

2.1 The Octave-Based Wrap (7-Note Scale)

Pythagoras began with a single note, say frequency f. He then multiplied by 3/2 repeatedly to generate new notes:

$$f, \quad f \cdot \frac{3}{2}, \quad f \cdot \left(\frac{3}{2}\right)^2, \quad f \cdot \left(\frac{3}{2}\right)^3, \quad \dots$$

But these notes quickly leave the original octave. To bring them back, we divide by powers of 2 (since 2:1 is an octave). For example:

$$f \cdot \left(\frac{3}{2}\right)^2 = f \cdot \frac{9}{4} \quad \longrightarrow \quad \frac{9}{8}f \quad \text{(divide by 2)}$$

After doing this for 7 steps, we get the familiar **7-note diatonic scale** (Do-Re-Mi-Fa-Sol-La-Ti). This is the original Pythagorean scale.

2.2 The Tritave-Based Wrap (11-Note Scale)

Instead of wrapping around the octave (2:1), what if we wrap around the **tritave**—the interval 3:1? This is natural because our generator is 3/2, and $3{:}1 = (3/2)^2 \cdot (2/1)^{-1}$, so it fits the same harmonic family.

Repeating the same process—stacking fifths and reducing modulo the tritave (3:1)—we generate an **11-note scale** within one tritave. This is the tritave analog of the 7-note octave scale.

2.3 Extended Wraps: 12 Notes (Octave) and 19 Notes (Tritave)

If we continue the octave wrap beyond 7 notes, after 12 fifths we get a scale very close to closing (off by the Pythagorean comma). This is the basis of the **12-note Pythagorean chromatic scale**.

Similarly, continuing the tritave wrap to 19 notes gives a scale that nearly closes within the tritave. Thus:

Octave system: 7-note (basic), 12-note (extended)

Tritave system: 11-note (basic), 19-note (extended)

But none of these close perfectly. Enter equal temperament.

3 Equal Temperament: A Perfect Circle of Notes

An **equal temperament** divides a repeating interval (like the octave or tritave) into N equal steps. Each step multiplies frequency by a fixed ratio:

For the octave: step ratio $= 2^{1/N}$

For the tritave: step ratio $= 3^{1/N}$

The magic is this: certain values of N make the equal-tempered fifth ($2^{k/N}$ for octave, or $3^{k/N}$ for tritave) *extremely close* to the true Pythagorean fifth (3/2).

Through calculation, we find two key temperaments:

53-TET (octave): 53 equal steps per octave. The fifth is $2^{31/53} \approx 1.49999$, only 0.00001 away from $3/2 = 1.5$.

84-TET (tritave): 84 equal steps per tritave. The fifth is $3^{49/84} \approx 1.50000$, equally astonishingly close.

Thus, 53-TET and 84-TET are near-perfect digital approximations of Pythagorean tuning—but they have too many notes for practical use. We need a way to *reduce* them.

Importantly, 53-TET and 84-TET are not the same scale—but they are dual perspectives on the same harmonic lattice. In 53-TET, the octave is perfect (2:1 exactly), while the tritave is slightly off. In 84-TET, the tritave is perfect (3:1 exactly), while the octave is slightly off. Both are excellent approximations of the Pythagorean spiral, and both support the same skip patterns.

4 Reduction by Skipping: The Key Insight

Here is the heart of this paper: we can select a subset of notes from a high-resolution temperament to recover the familiar 7-, 11-, 12-, and 19-note scales—using only a simple repeating pattern of skips.

4.1 The Basic Skip Pattern: 4–9–9–9

Start at note 0 in a temperament with enough resolution (e.g., 53-TET or 84-TET). Then skip forward by 4 steps, then 9, then 9, then 9, then repeat:

$$\text{Positions: } 0,\ 4,\ 13,\ 22,\ 31,\ 35,\ 44,\ 53 \equiv 0 \pmod{53}$$

This gives 7 notes before returning to the start—exactly the 7-note Pythagorean scale! The sequence of skips is:

$$\boxed{4,\ 9,\ 9,\ 9,\ 4,\ 9,\ 9}$$

Similarly, in 84-TET (tritave), the same pattern gives 11 notes:

$$0,\ 4,\ 13,\ 22,\ 31,\ 35,\ 44,\ 53,\ 62,\ 66,\ 75,\ 84 \equiv 0$$

Skip sequence: $\boxed{4,\ 9,\ 9,\ 9,\ 4,\ 9,\ 9,\ 9,\ 4,\ 9,\ 9}$

4.2 Refining the Pattern: Breaking 9 into 5 + 4

The number 9 can be split into $5 + 4$. If we replace every "9" in the above sequence with "5, 4", we get a finer-grained pattern:

Original: 4, $\underline{9}$, $\underline{9}$, $\underline{9}$, 4, ... Refined: 4, 5, 4, 5, 4, 5, 4, 4, 5, 4, 5, 4, ...

This new sequence selects more notes—specifically, 12 notes from octave-based systems and 19 from tritave-based systems.

For the octave:

$$\text{12-note skip sequence: } \boxed{4, 5, 4, 5, 4, 5, 4, 4, 5, 4, 5, 4}$$

For the tritave:

$$\text{19-note skip sequence: } \boxed{4, 5, 4, 5, 4, 5, 4, 4, 5, 4, 5, 4, 5, 4, 4, 5, 4, 5, 4}$$

Table 1: Skip Sequences and Resulting Scales

Base System	Skip Pattern	Notes Selected	Corresponds To
Octave-based	$4, 9, 9, 9, \ldots$	7	Pythagorean diatonic
Octave-based	$4, 5, 4, 5, \ldots$	12	Pythagorean chromatic
Tritave-based	$4, 9, 9, 9, \ldots$	11	Tritave diatonic
Tritave-based	$4, 5, 4, 5, \ldots$	19	Tritave chromatic

5 Designing Your Perfect Interval

This is where the system becomes truly powerful. The skip patterns described above do **not** depend on using 53-TET or 84-TET specifically. They work in *any* sufficiently fine equal temperament that approximates the Pythagorean spiral of fifths.

In fact, there exists a family of high-resolution temperaments, each designed to make **one specific interval perfectly exact**, while keeping all other Pythagorean intervals within 1–3 cents (well below the threshold of human perception). These include:

Perfect Octave: $2^{1/53} \to 53$ equal steps per octave

Perfect Tritave: $3^{1/84} \to 84$ equal steps per tritave

Perfect Fifth: $(3/2)^{1/31} \to 31$ equal steps per fifth

Perfect Fourth: $(4/3)^{1/22} \to 22$ equal steps per fourth

Perfect Major Second: $(9/8)^{1/9} \to 9$ equal steps per major second

Perfect Major Sixth: $(27/16)^{1/40} \to 40$ equal steps per major sixth

Perfect Minor Seventh: $(16/9)^{1/44} \to 44$ equal steps per minor seventh

Each of these defines a different "grid" of pitch. But because they all approximate the same underlying Pythagorean relationships, **the same skip sequences apply**.

For example:

Start with the 31-step-per-fifth temperament $(3/2)^{1/31}$. Apply the 4–9–9–9-4-9-9 skip pattern. You get a 7-note scale where the fifth is *perfectly* 3/2, and all other intervals (fourth, major second, etc.) are within 1.8 cents of their Pythagorean values.

Start with the 22-step-per-fourth temperament $(4/3)^{1/22}$. Apply the 4–5–4–5-4-5-4-4-5-4-5-4 pattern. You get a 12-note scale with a perfect fourth, and near-perfect fifths, thirds, and seconds.

The *relative skip pattern*—4, then 9, then 9, then 9—remains identical because it reflects the underlying harmonic ratios, not the absolute size of the step.

This means a composer or instrument designer can choose:

1. Which interval they want to be perfectly consonant (fifth, fourth, octave, etc.),
2. Whether they want a 7-, 11-, 12-, or 19-note scale,
3. Whether they prefer an octave- or tritave-based framework,

and then generate a scale that is both mathematically elegant and perceptually flawless.

6 Why "God's Scales"?

The title reflects a profound observation: the same simple skip patterns—$4, 9, 9, 9$ and its refinement $4, 5, 4, 5, 4, 5, 4$—generate harmonious scales in both the octave and tritave worlds,

and across a whole family of temperaments—each tuned to a different perfect interval. This symmetry suggests a universal structure underlying musical pitch, independent of human convention.

Moreover, the numbers 7, 12, 11, and 19 are not arbitrary:

7 and 12 arise from the best rational approximations to $\log_2 3$ (the number of octaves per fifth).

11 and 19 arise from approximations to $\log_3 2$ (the number of tritaves per fourth).

The skip sequences are the bridge between the infinite spiral of Pythagorean fifths and the finite circles of equal temperament. They are the algorithm that turns chaos into order—and now, with the ability to choose your perfect interval, they become a tool for intentional harmonic design.

7 Conclusion

We have shown that:

1. Pythagorean scales (7/12 notes in octave, 11/19 in tritave) are near-perfect subsets of high-resolution equal temperaments.

2. These subsets are selected by simple, repeating skip patterns: $4, 9, 9, 9$ and $4, 5, 4, 5, 4, 5, 4$.

3. The 9-step skip can be decomposed into 5+4, revealing a recursive structure.

4. Crucially, these patterns work across a family of temperaments—each making a different interval (octave, tritave, fifth, fourth, etc.) perfectly exact.

5. In every case, the resulting 7-, 11-, 12-, or 19-note scale has one mathematically pure interval and all others within 1–3 cents of Pythagorean ideals—inaudible deviation.

This framework unifies ancient theory and modern practice. It requires no advanced math—only the ability to count steps. And it offers a path to scales that are both mathematically elegant and musically expressive, tailored to the composer's harmonic intent.

As Pythagoras might have said: *All is number—but not all numbers are equal. Some are divine.*